Topological excitations in $N = 1$ Supersymmetric QFT.

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Abstract

The present paper deals with $N = 1$ 2D supersymmetric integrable quantum field theory. The S-matrix proposed to describe the interactions between supersymmetric particles is applied to theories involving topological excitations of zero central charge. Bound states can fit consistently within this type of theories, since the bootstrap can be shown to close. The topological character of the excitations and the similarity with the scattering of particles is fully understood when a kink sector is introduced in the theory.

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1 Introduction

In a resent work [1], based on the original work of Schoutens [2], the bootstrap equation was investigated in 1+1 integrable quantum field theories involving supersymmetric particles. The presence of a kink sector and its implications in such theories was considered as well. Particles were considered to fall into representations of $N = 1$ supersymmetry with zero central charge, while kinks carry non zero central charge. In addition, kinks are subject to an adjacency condition, reflecting the non trivial topology of the non-zero charge sector.

Schoutens has also discussed a very interesting case concerning excitations of a mixed character. They are carries of a SUSY representation with no central charge, but they also obey an adjacency condition. Despite the last crucial difference, the S-matrix proposed to describe the interactions between those excitations appears to be very similar with the one for particles. The present work is an attempt to explore theories involving such excitations and discover the origin of the above formal similarity.

This letter is organised as follows. In section 2 some aspects of supersymmetric theories involving particles are reviewed. Then Schoutens's theory for topological excitations is generalized to involve particles of different mass. Bootstrap is discussed in those theories as well. In section 3, a kink sector is introduced and the way it fits within theories of particles is shortly reviewed. The topological excitations are also considered in the precense of such a sector and their topological character is then explained. Finally, in section 4 the problem of diagonalizing the fermion parity is discussed.

2 Scattering of supersymmetric excitations of zero central charge

2.1 S-Matrix for super-particles

In 1+1 dimensions the basis of $N = 1$ SUSY irreducible representation with zero central charge consists of two states $\{|\phi(\theta)\rangle, |\psi(\theta)\rangle\}$ of mass m (θ is the rapidity of the particle). In this basis, the supercharges take the matrix form

$$Q = e^{\theta/2}\sqrt{m}\begin{pmatrix} 0 & \epsilon \\ \epsilon^* & 0 \end{pmatrix}, \quad \bar{Q} = e^{-\theta/2}\sqrt{m}\begin{pmatrix} 0 & \epsilon^* \\ \epsilon & 0 \end{pmatrix}, \quad Q_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

where $\epsilon = \exp(i\pi/4)$ and Q_L is the fermionic parity operartor. The action of supercharges on two-particle states $|A_1(\theta_1)A_2(\theta_2)\rangle = |A_1(\theta_1)\rangle \otimes |A_2(\theta_2)\rangle$ is

$$\Delta(Q) = Q \otimes I + Q_L \otimes Q, \quad \Delta(\bar{Q}) = \bar{Q} \otimes I + Q_L \otimes \bar{Q} \tag{2}$$

Under the assumption of integrability, it is possible to construct a minimal S-matrix $S_P(\theta)$ for a QFT involving superparticles in the above representation. This construction is based on the commutation with supercharges and the requirements for unitarity, crossing symmetry and the Yang-Baxter equation [2]. Bound states can be introduced by multiplying $S_P(\theta)$ with a purely bosonic consistent S-matrix $S_B(\theta)$ that does exhibit poles at particular imaginary values of rapidity difference [3]. The additional requirement for closing the bootstrap implies strong restrictions on the spectrum of the full theory described by $S_P(\theta) \otimes S_B(\theta)$.

The masses of the doublets have to be of the form

$$m_a = m\sin(a\pi/H), \ a = 1, 2, ..., n \tag{3}$$

where n is the total number of particles. If $H = 2n$, the particles are not self-conjugate. For simplicity, the particles will be taken to be self-conjugate. The fusing angles must also obey a specific rule:

$$u_{ab}^{a+b} = \frac{(a+b)\pi}{H} \ (a+b \leq n), \quad u_{ab}^{|b-a|} = \pi - \frac{|b-a|\pi}{H}, \ (a+b > n) \tag{4}$$

In this case the elements of $S_P(\theta)$ take the form

$$S_{\phi\phi\rightarrow\phi\phi}^{[ab]}(\theta) = \left(1 + \frac{2\sin(\frac{a+b}{2H}\pi)\cos(\frac{a-b}{2H}\pi)}{\sin(\frac{\theta}{i})}\right) g^{[ab]}(\theta),$$

$$S_{\psi\psi\rightarrow\psi\psi}^{[ab]}(\theta) = \left(-1 + \frac{2\sin(\frac{a+b}{2H}\pi)\cos(\frac{a-b}{2H}\pi)}{\sin(\frac{\theta}{i})}\right) g^{[ab]}(\theta)$$

$$S_{\phi\phi\rightarrow\psi\psi}^{[ab]}(\theta) = S_{\psi\psi\rightarrow\phi\phi}^{[ab]}(\theta) = \frac{\sqrt{\sin(\frac{a\pi}{H})\sin(\frac{b\pi}{H})}}{\cos(\frac{\theta}{2i})} g^{[ab]}(\theta),$$

$$ \tag{5}$$

$$S_{\phi\psi\rightarrow\phi\psi}^{[ab]}(\theta) = S_{\psi\phi\rightarrow\psi\phi}^{[ab]}(\theta) = \frac{\sqrt{\sin(\frac{a\pi}{H})\sin(\frac{b\pi}{H})}}{\sin(\frac{\theta}{2i})} g^{[ab]}(\theta),$$

$$S_{\phi\psi\rightarrow\psi\phi}^{[ab]}(\theta) = \left(1 - \frac{2\sin(\frac{a-b}{2H}\pi)\cos(\frac{a+b}{2H}\pi)}{\sin(\frac{\theta}{i})}\right) g^{[ab]}(\theta),$$

$$S_{\psi\phi\rightarrow\phi\psi}^{[ab]}(\theta) = \left(1 + \frac{2\sin(\frac{a-b}{2H}\pi)\cos(\frac{a+b}{2H}\pi)}{\sin(\frac{\theta}{i})}\right) g^{[ab]}(\theta).$$

The functions $g^{ab}(\theta)$ are fixed by unitarity and crossing symmetry:

$$g^{[ab]}(\theta) = R^{[ab]}(\theta)R^{[ab]}(i\pi - \theta),$$

$$R^{[ab]}(\theta) = \frac{1}{\Gamma(\frac{\theta}{2\pi i})\Gamma(\frac{\theta}{2\pi i} + \frac{1}{2})} \prod_{k=1}^{\infty} \frac{\Gamma(\frac{\theta}{2\pi i} + \frac{a+b}{2H} + k - 1)\Gamma(\frac{\theta}{2\pi i} - \frac{a+b}{2H} + k)}{\Gamma(\frac{\theta}{2\pi i} + \frac{a+b}{2H} + k - \frac{1}{2})\Gamma(\frac{\theta}{2\pi i} - \frac{a+b}{2H} + k + \frac{1}{2})}$$

$$\times \frac{\Gamma(\frac{\theta}{2\pi i} + \frac{a-b}{2H} + k - \frac{1}{2})\Gamma(\frac{\theta}{2\pi i} - \frac{a-b}{2H} + k - \frac{1}{2})}{\Gamma(\frac{\theta}{2\pi i} + \frac{a-b}{2H} + k)\Gamma(\frac{\theta}{2\pi i} - \frac{a-b}{2H} + k)}. \tag{6}$$

The fusion has the form

$$
\begin{aligned}
|\phi^a(\theta + i\bar{u}^{\bar{b}}_{a\bar{c}})\phi^b(\theta + i\bar{u}^{\bar{a}}_{b\bar{c}})\rangle &= (f_{\phi\phi})^c_{ab}|\phi^c(\theta)\rangle \\
|\psi^a(\theta + i\bar{u}^{\bar{b}}_{a\bar{c}})\psi^b(\theta + i\bar{u}^{\bar{a}}_{b\bar{c}})\rangle &= (f_{\psi\psi})^c_{ab}|\phi^c(\theta)\rangle \\
|\phi^a(\theta + i\bar{u}^{\bar{b}}_{a\bar{c}})\psi^b(\theta + i\bar{u}^{\bar{a}}_{b\bar{c}})\rangle &= (f_{\phi\psi})^c_{ab}|\psi^c(\theta)\rangle \\
|\psi^a(\theta + i\bar{u}^{\bar{b}}_{a\bar{c}})\phi^b(\theta + i\bar{u}^{\bar{a}}_{b\bar{c}})\rangle &= (f_{\psi\phi})^c_{ab}|\psi^c(\theta)\rangle
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
(f_{\phi\phi})^c_{ab} = \sqrt{S^{ab}_{\phi\phi\to\phi\phi}(iu^c_{ab})}, && (f_{\psi\psi})^c_{ab} = \sqrt{S^{ab}_{\psi\psi\to\psi\psi}(iu^c_{ab})} \\
(f_{\phi\psi})^c_{ab} = \sqrt{S^{ab}_{\phi\psi\to\psi\phi}(iu^c_{ab})}, && (f_{\psi\phi})^c_{ab} = \sqrt{S^{ab}_{\psi\phi\to\phi\psi}(iu^c_{ab})}
\end{aligned}
\tag{8}
$$

Notice at this stage that the same particle c can be represented equally well by *any* one of the b, c breathers

$$
\begin{aligned}
|\phi^c(\theta)\rangle &= \frac{1}{(f_{\phi\phi})^c_{ab}}|\phi^a(\theta + i\bar{u}^{\bar{b}}_{a\bar{c}})\phi^b(\theta + i\bar{u}^{\bar{a}}_{b\bar{c}})\rangle = \frac{1}{(f_{\psi\psi})^c_{ab}}|\psi^a(\theta + i\bar{u}^{\bar{b}}_{a\bar{c}})\psi^b(\theta + i\bar{u}^{\bar{a}}_{b\bar{c}})\rangle \\
&\tag{9} \\
|\psi^c(\theta)\rangle &= \frac{1}{(f_{\phi\psi})^c_{ab}}|\phi^a(\theta + i\bar{u}^{\bar{b}}_{a\bar{c}})\psi^b(\theta + i\bar{u}^{\bar{a}}_{b\bar{c}})\rangle = \frac{1}{(f_{\psi\phi})^c_{ab}}|\psi^a(\theta + i\bar{u}^{\bar{b}}_{a\bar{c}})\phi^b(\theta + i\bar{u}^{\bar{a}}_{b\bar{c}})\rangle
\end{aligned}
$$

Consistency of the bootstrap means that any one of the above expressions leads to the correct amplitute (5) when it replaces the corresponding particle in a scattering process.

2.2 S-matrix for topological excitations

The S-matrix constructed by Schoutens involves a set of four such excitations, all of equal mass m, interpolating between two vacua $1, 2$. They will be denoted as

$$
\{|B_{00}(\theta)\rangle, |B_{10}(\theta)\rangle, |B_{11}(\theta)\rangle, |B_{01}(\theta)\rangle\}
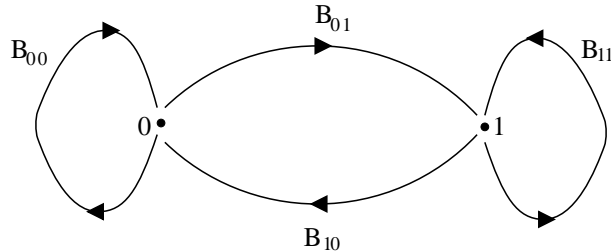$$

.

fig. 1: Topology of the four excitations

Obviously they form a reducible representation, consisting of two irreducible ones joined by the fermion parity:

$$
Q_L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{Q} = e^{\theta/2}\sqrt{m} \begin{pmatrix} 0 & e^{-i\pi/4} & 0 & 0 \\ e^{i\pi/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \\ 0 & 0 & e^{i\pi/4} & 0 \end{pmatrix}
$$

(10)

$$
\bar{\mathcal{Q}} = e^{-\theta/2}\sqrt{m} \begin{pmatrix} 0 & e^{i\pi/4} & 0 & 0 \\ e^{-i\pi/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \\ 0 & 0 & e^{-i\pi/4} & 0 \end{pmatrix}
$$

There are only eight admissable two-particle states. An S-matrix element for the process $B_{\alpha\beta}(\theta_1) + B_{\beta\gamma}(\theta_2) \to B_{\alpha\delta}(\theta_2) + B_{\delta\gamma}(\theta_1)$ will be denoted by

$$
S\left(\begin{matrix} \alpha & \delta \\ \beta & \gamma \end{matrix} \middle| \theta_1 - \theta_2 \right)
$$

The minimal expressions for the supersymmetric amplitudes, obeying unitarity, crossing symmetry and the Yang-Baxter equation, are:

$$
S\left(\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \middle| \theta \right) = \left(1 + \frac{2}{\sin(\theta/i)} \right) g(\theta)
$$

$$
S\left(\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \theta \right) = \left(-1 + \frac{2}{\sin(\theta/i)} \right) g(\theta)
$$

$$
S\left(\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \middle| \theta \right) = -\frac{1}{\cos(\theta/2i)} g(\theta)
$$

(11)

$$
S\left(\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \middle| \theta \right) = \frac{1}{\sin(\theta/2i)} g(\theta)
$$

$$
S\left(\begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \middle| \theta \right) = S\left(\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \middle| \theta \right) = g(\theta)
$$

where

$$
g(\theta) = R(\theta) R(i\pi - \theta), \quad R(\theta) = \frac{1}{\Gamma(\frac{\theta}{2\pi i})\Gamma(\frac{\theta}{2\pi i} + \frac{1}{2})} \prod_{k=1}^{\infty} \left(\frac{\Gamma(\frac{\theta}{2\pi i} - \frac{1}{2} + k)}{\Gamma(\frac{\theta}{2\pi i} + k)} \right)^4
$$

(12)

Crossing symmetry reads

$$
|\overline{B_{00}}(\theta)\rangle = |B_{00}(\theta)\rangle, \quad |\overline{B_{01}}(\theta)\rangle = |B_{10}(\theta)\rangle, \quad |\overline{B_{11}}(\theta)\rangle = |B_{11}(\theta)\rangle, \quad |\overline{B_{10}}(\theta)\rangle = |B_{01}(\theta)\rangle
$$

(13)

It can be easily seen that these amplitudes coincide with the ones in (5) for $a = b = H/2$. At this stage one wonders if, in the spirit of this similarity, the spectrum can be enlarged

to include supermultimpletes of different mass. The S-matrix elements for such a theory will be the same as in (5). The aim is to use these amplitudes in such a way that unitarity, crossing symmetry and the Young-Baxter are not spoiled by the presence of the adjacency condition.

Consider then n four-dimensional multipletes $\{|B_{00}^a(\theta)\rangle, |B_{10}^a(\theta)\rangle, |B_{11}^a(\theta)\rangle, |B_{01}^a(\theta)\rangle\}$ of mass m_a, $a = 1, 2, ..., n$. Keeping in advance an eye on the bootstrap , it is a natural choice for the spectrum to be of the form (3). It is not a hard guess to chose:

$$S^{[ab]}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}_{\phi\phi\to\phi\phi}(\theta)$$

$$S^{[ab]}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}(\theta)_{\psi\psi\to\psi\psi}$$

$$S^{[ab]}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\theta\end{pmatrix} = \begin{cases} S^{[ab]}(\theta)_{\phi\phi\to\psi\psi} \\ S^{[ab]}(\theta)_{\psi\psi\to\phi\phi} \end{cases}$$
(14)

$$S^{[ab]}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\theta\end{pmatrix} = \begin{cases} S^{[ab]}(\theta)_{\phi\psi\to\phi\psi} \\ S^{[ab]}(\theta)_{\psi\phi\to\psi\phi} \end{cases}$$

$$S^{[ab]}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}(\theta)_{\phi\psi\to\psi\phi}$$

$$S^{[ab]}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\theta\end{pmatrix} = S^{[ab]}(\theta)_{\psi\psi\to\phi\psi}$$

Unitarity and crossing symmetry can be clearly seen, while Yang-Baxter needs some more, but really straight forward calculation.

Naturally, the next step is to explore the possibility of introducing bound states by attaching a purely bosonic piece to the minimal supersymmetric S-matrix. The adjacency implies the following form for the fusion:

$$|B_{\alpha\beta}^a(\theta + iu_{a\bar{c}}^{\bar{b}})B_{\beta\gamma}^b(\theta - iu_{b\bar{c}}^{\bar{a}})\rangle = (f_{\alpha\beta\gamma}^{\alpha\gamma})_{ab}^c|B_{\alpha\gamma}^c(\theta)\rangle$$

where the greek indices label one of the two vacuua of the theory.

Notice that there are again two breathers corresponding to the same particle state:

$$|B_{\alpha\gamma}^c(\theta)\rangle = \frac{1}{(f_{\alpha\mu\gamma}^{\alpha\gamma})_{ab}^c}|B_{\alpha\mu}^a(\theta + iu_{a\bar{c}}^{\bar{b}})B_{\mu\gamma}^b(\theta - iu_{b\bar{c}}^{\bar{a}})\rangle, \quad \mu = 0, 1 \tag{15}$$

The fusion coupling are determined by the S-matrix elements:

$$(f_{\alpha\mu\gamma}^{\alpha\gamma})_{ab}^c = \sqrt{S^{[ab]}\begin{pmatrix} \alpha & \mu \\ \mu & \gamma \end{pmatrix} iu_{ab}^c\end{pmatrix}} \tag{16}$$

Working out the above relation one finds :

$$(f^{00}_{000})^c_{ab} = (f^{11}_{111})^c_{ab} = (f_{\phi\phi})^c_{ab} \quad , (f^{00}_{010})^c_{ab} = (f^{11}_{101})^c_{ab} = (f_{\psi\psi})^c_{ab}$$

$$(17)$$

$$(f^{01}_{001})^c_{ab} = (f^{10}_{110})^c_{ab} = (f_{\phi\psi})^c_{ab}, \quad (f^{01}_{011})^c_{ab} = (f^{10}_{100})^c_{ab} = (f_{\psi\phi})^c_{ab}$$

The bootsrap now reads

$$S^{[dc]} \begin{pmatrix} \alpha & \delta \\ \beta & \gamma \end{pmatrix} \theta_d - \theta_c \end{pmatrix} = \sum_{\nu=0,1} \frac{(f^{\alpha\delta}_{\alpha\nu\delta})^c_{ab}}{(f^{\beta\gamma}_{\beta\mu\gamma})^c_{ab}} S^{[da]} \begin{pmatrix} \alpha & \nu \\ \beta & \mu \end{pmatrix} \theta_d - \theta_a \end{pmatrix} S^{[db]} \begin{pmatrix} \nu & \delta \\ \mu & \gamma \end{pmatrix} \theta_d - \theta_b \end{pmatrix}$$

ie. $$(18)$$

$$S^{[dc]} \begin{pmatrix} \alpha & \delta \\ \beta & \gamma \end{pmatrix} \theta \end{pmatrix} = \sum_{\nu=0,1} \frac{(f^{\alpha\delta}_{\alpha\nu\delta})^c_{ab}}{(f^{\beta\gamma}_{\beta\mu\gamma})^c_{ab}} S^{[da]} \begin{pmatrix} \alpha & \nu \\ \beta & \mu \end{pmatrix} \theta - i\bar{u}^b_{ac} \end{pmatrix} S^{[db]} \begin{pmatrix} \nu & \delta \\ \mu & \gamma \end{pmatrix} \theta + i\bar{u}^a_{bc} \end{pmatrix}$$
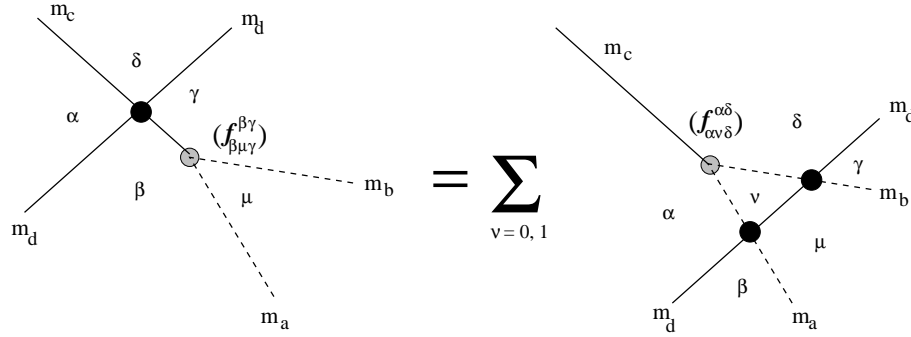

fig. 2: Bootstrap equation for topological excitations

Since the S-matrix elements involved are the same as the ones for particles, the bootstrap can be shown to close in the same fashion as in[1]. Hence we reach the conclusion that one can successfully build supersymetric theories involving excitations of zero central charge and non trivial topology. The adjacency condition is indeed compatible with the S-matrix elements (14). Further more, bound states can be introduced in a successful way that brings the adjacency condition in full consistency with the bootstrap.

3   Introducing the soliton sector

It is possible that the bosonic spectrum contains particles (or solitons) whose mass and fusing rules do not obey the crucial conditions (3), (4) respectively. A typical case is the Sine-Gordon theory, where the soliton and antisoliton are of equal mass m, while the particles (breathers) have mass

$$m_a = m \cos(\xi_a/2), \ a = 1, ..., n \ (\xi_a = \pi - 2a\pi/H) \tag{19}$$

The soliton can fuse with the antisoliton to any one of the particles, while soliton (antisoliton) fuse with any particle back to itself. The fusing angles are

$$u^a_{s\bar{s}} = u^a_{\bar{s}s} = \pi - \frac{2a\pi}{H}, \quad u^s_{sa} = u^s_{as} = u^{\bar{s}}_{\bar{s}a} = u^{\bar{s}}_{a\bar{s}} = \frac{\pi}{2} + \frac{a\pi}{H} \tag{20}$$

Clearly, such particles or solitons can not be consistently extended to two-dimensional multiplets of zero central charge. But it is possible to maintain their position in the spectrum of the supersymmetric theory if they are extended to supersymmetric kinks with non zero central charge [1], [4].

The soliton -antisoliton pair is extended to a set of four kinks interpolating between three vacua , labeled by $\{0, \frac{1}{2}, 1\}$. In the basis $\{|K_{0\frac{1}{2}}\rangle, |K_{1\frac{1}{2}}\rangle, |K_{\frac{1}{2}0}\rangle, |K_{\frac{1}{2}1}\rangle\}$ the action of the supercharges is realised [5], [2]:

$$Q = e^{\theta/2}\sqrt{m}\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \bar{Q} = e^{-\theta/2}\sqrt{m}\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$\tag{21}$$

The kinks $K_{0\frac{1}{2}}$ and $K_{1\frac{1}{2}}$ have $T = 1$, while their anti-kinks $K_{\frac{1}{2}0}$ and $K_{\frac{1}{2}1}$ have $T = -1$.
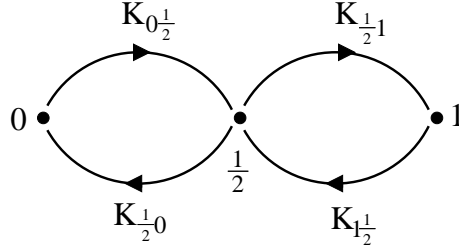

fig. 3: The four kinks interpolating between the three vacuua

The non-zero S-matrix elements are [6]

$$S\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\theta\end{pmatrix} = S\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\theta\end{pmatrix} = K(\theta)2^{(i\pi-\theta)/2\pi i}\cos\left(\frac{\theta}{4i} - \frac{\pi}{4}\right)$$

$$S\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\theta\end{pmatrix} = S\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix}\theta\end{pmatrix} = K(\theta)2^{\theta/2\pi i}\cos\left(\frac{\theta}{4i}\right)$$

$$S\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\theta\end{pmatrix} = S\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\theta\end{pmatrix} = K(\theta)2^{(i\pi-\theta)/2\pi i}\cos\left(\frac{\theta}{4i} + \frac{\pi}{4}\right) \tag{22}$$

$$S\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}\theta\end{pmatrix} = S\begin{pmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{pmatrix}\theta\end{pmatrix} = K(\theta)2^{\theta/2\pi i}\cos\left(\frac{\theta}{4i} - \frac{\pi}{2}\right)$$

The scalar function $K(\theta)$ is determined by crossing symmetry and unitarity. The minimal solution for $K(\theta)$ is

$$K(\theta) = \frac{1}{\sqrt{\pi}} \prod_{k=1}^{\infty} \frac{\Gamma(k - \frac{1}{2} + \theta/2\pi i)\Gamma(k - \theta/2\pi i)}{\Gamma(k + \theta/2\pi i)\Gamma(k + \frac{1}{2} - \theta/2\pi i)}.$$

There are six kink-antikink states which can form bound states at appropriate (imaginary) values of rapidity. For rapidity difference $\Delta\theta = iu_{s\bar{s}}^a$ these bound states are related to the particles $\{\phi^a(\theta), \psi^a(\theta)\}$:

$$|K_{\alpha\beta}(\theta + i\xi_a/2)K_{\beta\gamma}(\theta - i\xi_a/2)\rangle = (f_{\alpha\beta\gamma}^{\phi})_{ss}^a|\phi^a(\theta)\rangle + (f_{\alpha\beta\gamma}^{\psi})_{ss}^a|\psi^a(\theta)\rangle \tag{23}$$

The non-zero coupling constants are

$$f_{0\frac{1}{2}0}^{\phi} = f_{1\frac{1}{2}1}^{\phi} = 2^{(\pi-2\xi)/4\pi} f_{\frac{1}{2}0\frac{1}{2}}^{\phi} = 2^{(\pi-2\xi)/4\pi} f_{\frac{1}{2}1\frac{1}{2}}^{\phi} = \sqrt{K(i\xi)2^{(\pi-\xi)/2\pi} \cos\left(\frac{\xi - \pi}{4}\right)}$$

$$f_{1\frac{1}{2}0}^{\psi} = -f_{0\frac{1}{2}1}^{\psi} = 2^{(\pi-2\xi)/4\pi} i f_{\frac{1}{2}0\frac{1}{2}}^{\psi} = -2^{(\pi-2\xi)/4\pi} i f_{\frac{1}{2}1\frac{1}{2}}^{\psi} = \sqrt{K(i\xi)2^{(\pi-\xi)/2\pi} \cos\left(\frac{\xi + \pi}{4}\right)}.$$

The picture of the particles $\{|\phi^a(\theta)\rangle, |\psi^a(\theta)\rangle\}$ in terms of the breathers deserves of special attention. The breathers are still subject to the adjacency condition, while the particles are not. Hence, one needs *a set* of breathers to represent a particle. In the Hilbert space formalism this can be achieved through a summation:

$$|\phi^a(\theta)\rangle = \sum_{\alpha,\beta,\gamma} \frac{1}{(f_{\alpha\beta\gamma}^{\phi})_{ss}^a} |K_{\alpha\beta}(\theta + i\xi_a/2)K_{\beta\gamma}(\theta - i\xi_a/2)\rangle$$

$$|\psi^a(\theta)\rangle = \sum_{\alpha,\beta,\gamma} \frac{1}{(f_{\alpha\beta\gamma}^{\psi})_{ss}^a} |K_{\alpha\beta}(\theta + i\xi_a/2)K_{\beta\gamma}(\theta - i\xi_a/2)\rangle \tag{24}$$

where the parameters α, β, γ can take the values $0, \frac{1}{2}, 1$ in a way that respects the topology of the kinks. In fact each one of the particles needs four breathers to be represented. It was first shown in [6] that the breather scattering leads to the correct amplitudes (5) for the particles, ie the bootstrap closes.

In the case of the zero charge topological excitations, the states $\{|B_{\alpha\gamma}^a(\theta)\rangle, \alpha, \gamma = 0, 1\}$ do obey an adjacency condition. It is now obvious that when the solitons are included in the spectrum, each one of the above states can be identified (up to a fusion constant) with a breather:

$$|K_{\alpha\beta}(\theta + i\xi_a/2)K_{\beta\gamma}(\theta - i\xi_a/2)\rangle = (f_{\alpha\beta\gamma}^{\phi})_{ss}^a|B_{\alpha\gamma}^a(\theta)\rangle \tag{25}$$

where α, γ can only take the values $0, 1$.

So, in the presence of a soliton sector, the the mixed character of the quantum excitations $|B_{\alpha\beta}^a(\theta)\rangle$ can be cery well understood in terms of kink-antikink breathers. The fact that particles are represented by a set of the same breathers also explains the formal similarity of the amplitudes.

4 Adjacency condition and fermion parity

In the precence of a non trivial topology, states carry topological indices related to the pair of vacua that the excitation interpolates between. It is this pair that actually changes under the action of supercharges, hence the fermionic degree is incoded within it. Notice also that the adjacency condition forces fermion parity to a non diagonal form. When particles are considered, the states are represented by a set of topological breathers, such as adjacency condition is terminated:

$$|\phi^a(\theta)\rangle = \frac{1}{\sqrt{2}}(|B_{00}^a(\theta)\rangle + |B_{11}^a(\theta)\rangle), \quad |\psi^a(\theta)\rangle = \frac{1}{\sqrt{2}}(|B_{01}^a(\theta)\rangle + |B_{10}^a(\theta)\rangle) \tag{26}$$

In this case, the fermionic degree is released from its topological character and fermion parity is diagonalized. It would be very interesting if one could apply the same idea in the case of solitons. Remember that semiclassical analysis suggests such a picture for supersolitons [9], since they appear to be fermion number eigenstates with fractional eigenvalues . In the spirit of diagonalising the fermion parity, the following summation can be attempted for the states of the kink states:

$$|u(\theta)\rangle = \frac{1}{\sqrt{2}}(|K_{0\frac{1}{2}}(\theta)\rangle + |K_{1\frac{1}{2}}(\theta)\rangle), \qquad |\bar{u}(\theta)\rangle = \frac{1}{\sqrt{2}}(|K_{\frac{1}{2}0}(\theta)\rangle + |K_{\frac{1}{2}1}(\theta)\rangle)$$

$$\tag{27}$$

$$|d(\theta)\rangle = \frac{1}{\sqrt{2}}(|K_{0\frac{1}{2}}(\theta)\rangle - |K_{1\frac{1}{2}}(\theta)\rangle), \qquad |\bar{d}(\theta)\rangle = \frac{1}{\sqrt{2}}(|K_{\frac{1}{2}0}(\theta)\rangle - |K_{\frac{1}{2}1}(\theta)\rangle)$$

The adjacency condition has not completely dissappeared, but it now defines a different topology for the states. The are only two vacua, A (coming from $0, 1$) and B ($\frac{1}{2}$). States of the same charge interpolate now between the same vacua and transform one to each other under SUSY:

$$\mathcal{Q}|u(\theta)\rangle = -ie^{\theta/2}|d(\theta) \qquad \mathcal{Q}|\bar{u}(\theta)\rangle = e^{\theta/2}|\bar{d}(\theta)\rangle \tag{28}$$

$$\mathcal{Q}|d(\theta)\rangle = +ie^{\theta/2}|u(\theta) \qquad \mathcal{Q}|\bar{d}(\theta)\rangle = e^{\theta/2}|\bar{u}(\theta)\rangle \tag{29}$$

So, in this basis we have achieved a local action for the supercharges and released the fermionic number from the topology.

The problem is that the above change of basis is not compatible with crossing symmetry [2]. I have attempted to reconstruct a supersymmetric soliton S-matrix for the states of type (27). Crossing symmetry forces one to extend the algebra to $N = 2$ SUSY in order to gain consistent braiding for the action on multi-soliton states. Of course there is no problem when $T = 0$ (section 2). This generates the suspicion that the pathology of the basis (27) is somehow related with the reducibility of the corresponding $N = 1$ representation.

Nevertheless, it seems after all that one is obliged to work with the initial, "non-diagonal" basis. The next step is to gain a realization for the fermion number within
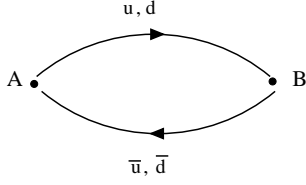
fig. 4: Topology of the kinks when diagonalise fermion parity

this basis. Perhaps the mechanism for representing a soliton by a set of two kinks is not wrong. It could be simply that the summation rule is not the right way of applying the idea. The problem will be investigated extensiveli in future work.

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